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Let figure 2 represent the force diagram, $ab=W$, $bc=R$, $ac=R_1$, $\angle abc=90^\circ-\theta$, $\angle bac=90^\circ-\varphi$, $\angle acb=\theta+\varphi$.

$\therefore$ $R=W\cos\varphi/\sin(\theta+\varphi)$, $R_1=W\cos\theta/\sin(\theta+\varphi)$.

$\therefore$ $R=5.5078$ pounds, $R_1=7.1881$ pounds.

DIOPHANTINE ANALYSIS.

61. Proposed by SYLVESTER ROBBINS, North Branch Depot, New Jersey.

Investigate that infinite series of prime, integral, rational scalene triangles where the sides of every term are consecutive numbers; then take the necessary factors from the proper KEY, and by an expeditious method, find in their order the areas of ten initial terms.

Solution by the PROPOSER.

I. The KEY to this series of rational triangles is $\sqrt{3}=\frac{1}{1}, \frac{2}{1}, \frac{5}{3}, \frac{7}{4}, \frac{19}{11}, \frac{26}{15}, \frac{71}{41}, \frac{97}{56}, \frac{265}{153}, \frac{362}{209}, \frac{989}{571}, \frac{1351}{780}, \frac{3691}{2131}, \frac{5042}{2911}, \frac{13775}{7953}, \frac{18817}{10864}, \frac{51409}{29681}, \frac{70226}{40545}, \frac{191861}{110771}, \frac{262087}{151316}$, etc. Regard the mean side as the base, and drop perpendicular from the opposite angle. Let $x=\frac{1}{2}$base. Notice that $x-2$ and $x+2$ are the segments of the base, and $\sqrt{[3(x^2-1^2)]}$ is the altitude of the triangle. Find such values for x as will render $\sqrt{[3(x^2-1^2)]}$ rational.

When $x=1$, 2, 7, 26, 97, 362, 1351, 5042, 18817, 70226, 262087, $\sqrt{[3(x^2-1^2)]}=0$, 3, 12, 45, 168, 627, 2320, 8733, 32592, 121635, 453948, etc.

These values of x are the half-bases of the several triangles. They are also the numerators of the even convergents in the expansion of $\sqrt{3}$. The values of $\sqrt{[3(x^2-1^2)]}$ are the altitudes of the same triangles, respectively, and they are also three times the denominators of the even convergents in the expansion of $\sqrt{3}$. Multiply one-half the base of a triangle by its perpendicular height, or, three times the product of the terms of the nth even convergent, must give the area of the nth triangle in the series.

Thus, $3\times2\times1=6$; $3\times7\times4=84$; $3\times26\times15=1170$; $3\times97\times56=16296$; $3\times362\times209=226974$; $3\times1351\times780=3161340$; $3\times5042\times2911=44031786$; $3\times18817\times10864=613283664$; $3\times70226\times40545=8541939510$; $3\times262087\times151316=118973869476$, etc.

II. Numerators of even convergents in expansion of $\sqrt{3}$: 1, 2, 7, 26, 97, 362, 1351, 5042, 18817, 70226, etc. Then $\frac{1}{8}(7^2-1^2)=6$; $\frac{1}{8}(26^2-2^2)=84$; $\frac{1}{8}(97^2-7^2)=1170$; $\frac{1}{8}(362^2-26^2)=16296$; $\frac{1}{8}(1351^2-97^2)=226974$; etc.

III. Denominators of even convergents : 1, 4, 15, 56, 209, 780, 2911, etc. $\frac{3}{8}(4^2-0)=6$; $\frac{3}{8}(15^2-1^2)=84$; $\frac{3}{8}(56^2-4^2)=1170$; $\frac{3}{8}(209^2-15^2)=16296$; $\frac{3}{8}(780^2-56^2)=226974$; etc.

IV. Let $x=$the half-sum of the three sides of the triangle. Then $\frac{1}{3}x-1$, $\frac{1}{3}x$ and $\frac{1}{3}x+1$ are the remainders.

$(x)[(\frac{1}{3}x)-1](\frac{1}{3}x)[(\frac{1}{3}x)+1]=$square of triangle. $3(x^2-3^2)=$square.

$\sqrt{\{(\frac{1}{3}x^2)[(x^2-3^2)/9]\}}/x=\frac{1}{3}\sqrt{[(x^2-3^2)/3]}$, the radius of inscribed circle.

Put $x=y+6$; then $3[(y+6)^2-3^2]=$square$=(my+9)^2$.

$3y^2+36y+81=m^2y^2+18my+81$; $3y+36=m^2y+18m$. $y=(18m-36)/(3-m^2)$; and $x=y+6=(18m-18-6m^2)/(3-m^2)$.

Say $m=\sqrt{3}=\frac{1}{1}, \frac{2}{1}, \frac{5}{3}, \frac{7}{4}, \frac{19}{11}, \frac{26}{15}$, etc. Substitute these values of m in the formula, and we obtain, when

$m=1, x=3$, and $\frac{1}{3}\sqrt{[x^2-3^2)/3]}=0$;
$m=2, x=6$, “ “ $=1$;
$m=\frac{5}{3}, x=21$, “ “ $=4$;
$m=\frac{7}{4}, x=78$, “ “ $=15$;
$m=\frac{19}{11}, x=291$, “ “ $=56$;
$m=\frac{26}{15}, x=1086$, “ “ $=209$; etc.

Here the several values of x, the half-sums of three sides, are also three times the numerators of the even convergents in the expansion of $\sqrt{3}$, and the several values of $\frac{1}{3}\sqrt{[(x^2-3^2)/3]}$ are the radii of the inscribed circles, respectively, and also the denominators of the even convergents. Multiplying the half-sum by the radius of the inscribed circle, or taking three times the product of the terms of the nth even convergent in the expansion of $\sqrt{3}$ will give the area of the nth triangle in the series where the common difference in sides is unity.

V. Denominators of odd convergents : 1, 3, 11, 41, 153, 571, 2131, etc. $\frac{3}{4}(3^2-1^2)=6$; $\frac{3}{4}(11^2-3^2)=84$; $\frac{3}{4}(41^2-11^2)=1170$; $\frac{3}{4}(153^2-41^2)=16296$; $\frac{3}{4}(571^2-153^2)=226974$; etc.

VI. Numerators of odd convergents : 1, 5, 19, 71, 265, 989, 3691, etc. $(2\frac{1}{2})^2-(\frac{1}{2})^2=6$; $(9\frac{1}{2})^2-(2\frac{1}{2})^2=84$; $(35\frac{1}{2})^2-(9\frac{1}{2})^2=1170$; $(132\frac{1}{2})^2-(35\frac{1}{2})^2=16296$; $(494\frac{1}{2})^2-(132\frac{1}{2})^2=226974$; etc.

VII. Let $x+1$, x and $x-1$ be the remainders. Then $3x$=the half-sum. $(3x)(x+1)(x)(x-1)$=square of area. $3(x^2-1^2)$=square. Put $x=y+2$. $3[(y+2)^2-1^2]$=square$=(my+3)^2$. $3y^2+12y+9=m^2y^2+6my+9$; $y=(6m-12)/(3-m^2)$; $x=(6m-6-2m^2)/(3-m^2)$. Consent that $m=\sqrt{3}=\frac{1}{1}, \frac{2}{1}, \frac{5}{3}, \frac{7}{4}, \frac{19}{11}$, etc. Substitute in formula, and $x=1, 2, 7, 26, 97, 362, 1351$, etc., and the areas of the triangles are as follows : $\sqrt{(3\times 2\times 1\times 0)}=3\times 1\times 0=0$; $\sqrt{(6\times 3\times 2\times 1)}=3\times 2\times 1=6$; $\sqrt{(21\times 8\times 7\times 6)}=3\times 7\times 4=84$; $\sqrt{(18\times 27\times 26\times 25)}=3\times 26\times 15=1170$; $\sqrt{(291\times 98\times 97\times 96)}=3\times 97\times 56=16296$; $\sqrt{(1086\times 363\times 362\times 361}=3\times 362\times 209=226974$, etc.

VIII. Odd convergents : $\sqrt{3}=\frac{1}{1}, \frac{5}{3}, \frac{19}{11}, \frac{71}{41}, \frac{265}{153}$, etc. $6\times 1\times 1=6$; $(6\times 3\times 5)-6=84$; $(6\times 11\times 19)-84=1170$; $(6\times 41\times 71)-1170=16296$; $(6\times 153\times 265)-16296=226974$; etc.

IX. Relation of areas : $14\triangle_n-\triangle_{n-1}=\triangle_{n+1}$; $M=14$. 6 ; $14\times 6=84$; $(14\times 84)-6=1170$; $(14\times 1170)-84=16296$; $(14\times 16296)-1170=226974$, etc.

X. When x represents the half-base, $2x-1$, $2x$ and $2r+1$ are the sides, $x-2$ and $x+2$ are the segments of the base , $\sqrt{[3(x^2-1^2)}$ is the altitude of the triangle ; and $r\sqrt{[3(x^2-1^2)]}/(4x^2-1^2)$ is the half-sine of angle opposite base. Giving x same values as in I, the half-sines are found to be $\frac{2}{5}=(2\times 1)/(2^2+1^2)$; $\frac{28}{65}=(7\times 4)/(7^2+4^2)$; $\frac{390}{901}=(26\times 15)/(26^2+15^2)$; $\frac{5432}{12545}=(97\times 56)/(97^2+56^2)$; $75658/(362^2+209^2)=(362\times 209)/(362^2+209^2)$; etc.

Multiply the product of two sides by one-half sine of included angle for area of triangle.

$[(2\times 2)-1][(2\times 2)+1][(2\times 1)/(2^2+1^2)]=3\times 5\times\frac{2}{5}=3\times 1\times 2=6$; $[(2\times 7)-1]$

$[(2\times7)+1][(4\times7)/(4^2+7^2)]=13\times15\times\frac{28}{65}=3\times4\times7=84$; $[(2\times26)-1][(2\times26+1]$ $[(15\times26)/(15^2+26^2)]=51\times53\times\frac{390}{901}=3\times15\times26=1170$; $[(2\times97)-1][(2\times97+1]$ $[(56\times97)/(56^2+97^2)]=193\times195\times\frac{5432}{12545}=3\times56\times97=16296$; $[(2\times362)-1][(2\times362)+1]$ $[(209\times362)/(209^2+362^2)]=723\times725\times\frac{75658}{174725}=3\times209\times362=226974$; etc.

Here it should be noticed that in canceling both sides the denominator of the half-sine disappears, and three times the product of the terms of the nth even convergent in the expansion of $\sqrt{3}$ brings the area to light ; also observe, since sides and denominator fall out of view, and factor 3 stands constant, the area must be determined by the numerator of these half-sines, and *this* series may be continued by use of Magic $M=14$; $14\times2=28$; $(14\times28)-2=390$; $(14\times390)-28=5432$; $(14\times5432)-390=75658$; etc.

Also solved by *JOSIAH H. DRUMMOND*, *M. A. GRUBER*, and *G. B. M. ZERR.*

AVERAGE AND PROBABILITY.

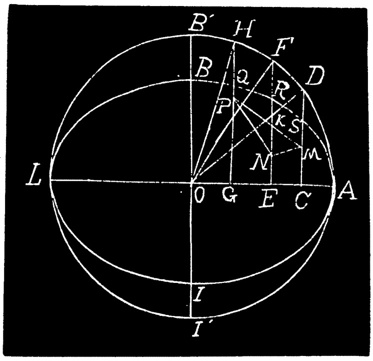
60. Proposed by **B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.**

Four points are taken at random within an ellipse. What is the chance that they form a reentrant quadrilateral ?

Solution by **G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.**

We will solve this problem for the quadrant, the semi-ellipse, and the whole ellipse.

Let $ABLI$ be the ellipse, and $AB'LI'$ the circumscribing circle ; M, N, P the three random points ; through M, N, P draw CD, EF, GH perpendicular to AO, EF intersecting MP at K. The triangle will pass through all the possible variations by considering only those relative positions of the points in which CD lies to the right of GH, and EF between CD and GH.



If the fourth point falls anywhere on the triangle formed by joining the points M, N, P, the quadrilateral thus formed will be reentrant.

Let $OA=a$, $OB=b$, $GP=x$, $CM=y$, $EN=z$, $GQ=x'$, $CS=y'$, $ER=z'$, $EK=z''$, $\angle GOH=\theta$, $\angle COD=\varphi$, $\angle EOF=\psi$.

Then we have $x'=b\sin\theta$, $y'=b\sin\varphi$, $z'=b\sin\psi$, $v=1/(\cos\varphi-\cos\theta)$, $z''=v[x(\cos\varphi-\cos\psi)+y(\cos\psi-\cos\theta)]$.

Area $MNP=\frac{1}{2}a[x(\cos\varphi-\cos\psi)+y(\cos\psi-\cos\theta)+z(\cos\theta-\cos\varphi)]=u$, when $z<z''$. Area $MNP=\frac{1}{2}a[x(\cos\psi-\cos\varphi)+y(\cos\theta-\cos\psi)+z(\cos\varphi-\cos\theta)]=u_1$, when